ZONE DIAGRAMS IN EUCLIDEAN SPACES AND IN OTHER NORMED SPACES

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Abstract. Zone diagram is a variation on the classical concept of a Voronoi diagram. Given n sites in a metric space that compete for territory, the zone diagram is an equilibrium state in the competition. Formally it is defined as a fixed point of a certain "dominance" map.

Asano, Matoušek, and Tokuyama proved the existence and uniqueness of a zone diagram for point sites in Euclidean plane, and Reem and Reich showed existence for two arbitrary sites in an arbitrary metric space. We establish existence and uniqueness for n disjoint compact sites in a Euclidean space of arbitrary (finite) dimension, and more generally, in a finite-dimensional normed space with a smooth and rotund norm. The proof is considerably simpler than that of Asano et al. We also provide an example of non-uniqueness for a norm that is rotund but not smooth. Finally, we prove existence and uniqueness for two point sites in the plane with a smooth (but not necessarily rotund) norm.

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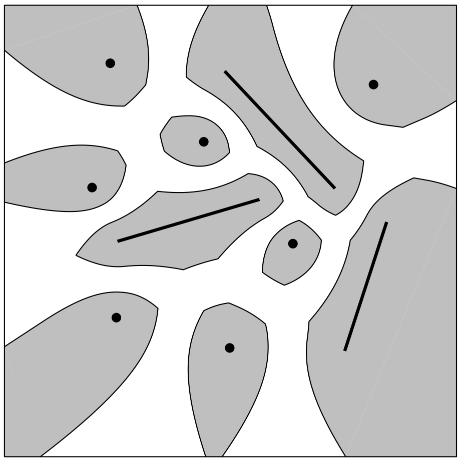

FIGURE 1. A zone diagram of points and segments.

1. Introduction

Zone diagram is a metric notion somewhat similar to the classical concept of a Voronoi diagram. Let (X, dist) be a metric space and let $\mathbf{P} = (P_1, \ldots, P_n)$ be an n-tuple of nonempty subsets of X called the *sites*. To avoid unpleasant trivialities, we will always assume in this paper that the sites are closed and pairwise disjoint.

A *zone diagram* of the n-tuple $\mathbf{P}$ is an n-tuple $\mathbf{R} = (R_1, \ldots, R_n)$ of subsets of X, called the *regions* of the zone diagram, with the following defining property: Each R_i consists of all points $x \in X$ that are closer (non-strictly) to P_i than to the union $\bigcup_{j \neq i} R_j$ of all the other regions.

Fig. 1 shows a zone diagram in Euclidean plane whose sites are points and segments. While in the Voronoi diagram the regions partition the whole space, in a zone diagram the union of the regions typically has a nonempty complement, called the *neutral zone*.

The definition of the zone diagram is implicit, since each region is determined in terms of the remaining ones. So neither existence nor uniqueness of the zone diagram is obvious, and so far only partial results in this direction have been known.

Asano et al. [2] introduced the notion of a zone diagram, for the case of n point sites in Euclidean plane, and in this setting they proved existence and uniqueness. The proof involves a case analysis specific to $\mathbb{R}^2$.

Reem and Reich [8] established, by a simple and elegant argument, the existence of a zone diagram for *two* sites in an arbitrary metric space (and even in a still more general setting, which they call *m-spaces*).

On the negative side, they gave an example of a three-point metric space in which the zone diagram of two point sites is not unique; thus, *uniqueness* needs additional assumptions. On the other hand, for all we know, it is possible that a zone diagram always exists, for arbitrary sites in an arbitrary metric space.

Arbitrary sites in Euclidean spaces. In this paper, we establish the existence and uniqueness of zone diagrams in Euclidean spaces. This generalizes the main result of [2] with a considerably simpler argument. For the case of two point sites in the plane, we also obtain a new and simpler proof of the existence and uniqueness of the *distance trisector curve* considered by Asano et al. [3].

Theorem 1.1. *Let the considered metric space (X, dist) be $\mathbb{R}^d$ with the Euclidean distance. For every n-tuple $\mathbf{P} = (P_1, \ldots, P_n)$ of nonempty closed sites in $\mathbb{R}^d$ such that $\mathrm{dist}(P_i, P_j) > 0$ for every $i \neq j$, there exists exactly one zone diagram $\mathbf{R}$.*

The full proof is contained in Sections 2 (general preliminaries) and 3. The same proof yields existence and uniqueness also for infinitely many sites in $\mathbb{R}^d$, provided that every two of them have distance at least 1 (or some fixed $\varepsilon > 0$). Moreover, with some extra effort it may be

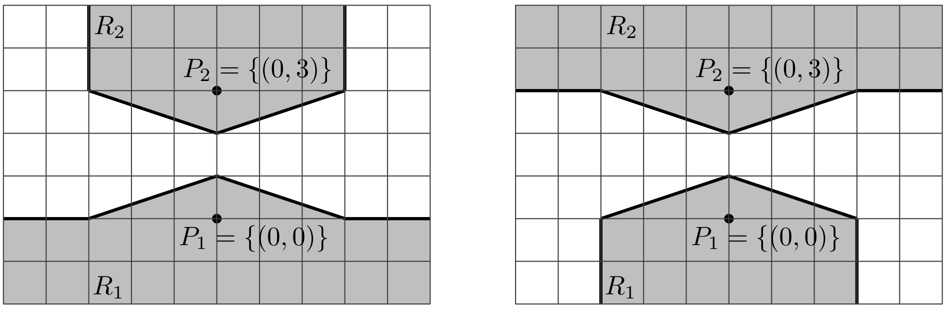

FIGURE 2. Two different zone diagrams under the ℓ_1 metric (drawn in the grid with unit spacing).

possible to extend the proof to compact sites in a Hilbert space, for example, but in this paper we restrict ourselves to the finite-dimensional setting.

Normed spaces. We also investigate zone diagrams in a more general class of metric spaces, namely, finite-dimensional normed spaces.[3] Normed spaces are among the most important classes of metric spaces. Moreover, as we will see, studying arbitrary norms also sheds some light on the Euclidean case. Earlier Asano and Kirkpatrick [1] investigated distance trisector curves (which are essentially equivalent to two-site zone diagrams) of two point sites under polygonal norms in the plane, obtaining results for the Euclidean case through approximation arguments.

For us, a crucial observation is that the uniqueness of zone diagrams does *not* hold for normed spaces. Let us consider $\mathbb{R}^2$ with the ℓ_1 norm $\|\cdot\|_1$, given by $\|x\|_1 = |x_1| + |x_2|$. It is easy to check that the two point sites $(0,0)$ and $(0,3)$ have at least two different zone diagrams, as drawn in Fig. 2. This example was essentially contained already in Asano and Kirkpatrick [1], although in a different context.

The ℓ_1 norm differs from the Euclidean norm in two basic respects: the unit ball has sharp corners and straight edges; in other words, the ℓ_1 norm is neither smooth nor rotund. We recall that a norm $\|\cdot\|$ on $\mathbb{R}^d$ is called *smooth* if the function $x \mapsto \|x\|$ is differentiable (geometrically, the unit ball of a smooth norm has no "sharp corners"; see Fig. 3).[4] A norm $\|\cdot\|$ on $\mathbb{R}^d$ is called *rotund* (or *strictly convex*) if for all $x, y \in \mathbb{R}^d$ with $\|x\| = \|y\| = 1$ and $x \neq y$ we have $\|\frac{x+y}{2}\| < 1$. Geometrically, the unit sphere of $\|\cdot\|$ contains no segment. By compactness, a rotund norm on a finite-dimensional space is also *uniformly convex*, which means that for every $\varepsilon > 0$ there is $\mu = \mu(\varepsilon) > 0$ such that if x, y are unit vectors with $\|x - y\| \geq \varepsilon$, then

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \mu$$

(we refer to [5] for this and other facts on norms mentioned without proofs).

The Euclidean norm $\|\cdot\|_2$, and more generally, the ℓ_p norms with $1 < p < \infty$, are both rotund and smooth. We have the following generalization of Theorem 1.1:

Theorem 1.2. *Let the considered metric space* (X, dist) *be* $\mathbb{R}^d$ *with a norm* $\|\cdot\|$ *that is both smooth and rotund. For every* n*-tuple* $\mathbf{P} = (P_1, \ldots, P_n)$ *of nonempty closed sites in* $\mathbb{R}^d$ *such that* $\mathrm{dist}(P_i, P_j) > 0$ *for every* $i \neq j$*, there exists exactly one zone diagram* $\mathbf{R}$.

[3] A finite-dimensional (real) normed space can be thought of as the real vector space $\mathbb{R}^d$ with some *norm*, which is a mapping that assigns a nonnegative real number $\|x\|$ to each $x \in \mathbb{R}^d$ so that $\|x\| = 0$ implies $x = 0$, $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $\alpha \in \mathbb{R}$, and the triangle inequality holds: $\|x + y\| \leq \|x\| + \|y\|$. Each norm $\|\cdot\|$ defines a metric by $\mathrm{dist}(x, y) := \|x - y\|$.

For studying a norm $\|\cdot\|$, it is usually good to look at its *unit ball* $\{\, x \in \mathbb{R}^d : \|x\| \leq 1 \,\}$. The unit ball of any norm is a closed convex body K that is symmetric about 0 and contains 0 in the interior. Conversely, any $K \subset \mathbb{R}^d$ with the listed properties is the unit ball of a (uniquely determined) norm.

[4] There are several notions of differentiability of functions on Banach spaces, such as the existence of directional derivatives, Gâteaux differentiability, Fréchet differentiability, or uniform Fréchet differentiability. However, in finite-dimensional Banach spaces they are all equivalent.

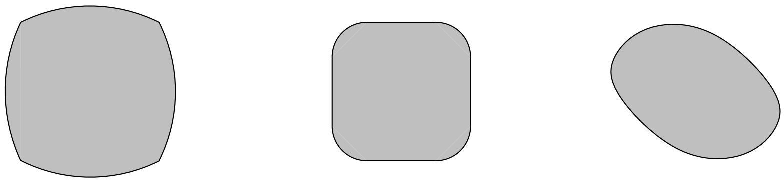

FIGURE 3. Rotundity and smoothness of norms.

The proof for the Euclidean case, i.e., of Theorem 1.1, is set up so that it generalizes to smooth and rotund norms more or less immediately; there is only one lemma where we need to work harder—see Section 4.

Our current proof method apparently depends both on smoothness and on rotundity. In Section 5 we show that smoothness is indeed essential, by exhibiting a non-smooth but rotund norm in $\mathbb{R}^d$ with non-unique zone diagrams. On the other hand, we suspect that the assumption of rotundity in Theorem 1.2 can be dropped. Currently we have a proof (see Appendix A) only in a rather special case:

Theorem 1.3. *For two point sites $P_0 = \{p_0\}$ and $P_1 = \{p_1\}$ in the plane $\mathbb{R}^2$ with a smooth norm, there exists exactly one zone diagram.*

2. Preliminaries

Here we introduce notation and present some results from the literature, some of them in a more general context than in the original publications.

Let (X, dist) be a general metric space. The closure of a set $A \subseteq X$ is denoted by $\overline{A}$, while ∂A stands for its boundary. The (closed) ball of radius r centered at x is denoted by $\mathrm{B}(x, r)$.

For sets $A, B \subseteq X$, not both empty, we define the *dominance region* of A over B as the set

$$\mathrm{dom}(A, B) := \{\, x \in X : \mathrm{dist}(x, A) \leq \mathrm{dist}(x, B) \,\},$$

where

$$\mathrm{dist}(C, D) := \inf_{x \in C,\ y \in D} \mathrm{dist}(x, y) \in [0, +\infty]$$

denotes the distance of sets C and D.

Let us fix an n-tuple $\mathbf{P} = (P_1, \ldots, P_n)$ of sites, i.e., nonempty subsets of X (which, as above, we assume to be disjoint and closed). For an n-tuple $\mathbf{R} = (R_1, \ldots, R_n)$ of arbitrary subsets of X, we define another n-tuple of regions $\mathbf{R}' = (R'_1, \ldots, R'_n)$ denoted by $\mathbf{Dom}\,\mathbf{R}$ and given by

$$R'_i := \mathrm{dom}\Big(P_i, \bigcup_{j \neq i} R_j\Big), \qquad i = 1, \ldots, n$$

(the sites are considered fixed and they are a part of the definition of the operator **Dom**).

The definition of a zone diagram can now be expressed as follows: An n-tuple $\mathbf{R}$ is called a *zone diagram* for the n-tuple $\mathbf{P}$ of sites if $\mathbf{R} = \mathbf{Dom}\,\mathbf{R}$ (componentwise equality, i.e., $R_i = \mathrm{dom}\big(P_i, \bigcup_{j \neq i} R_j\big)$ for all i).

For two n-tuples $\mathbf{R}$ and $\mathbf{S}$ of sets, we write $\mathbf{R} \preceq \mathbf{S}$ if $R_i \subseteq S_i$ for every i. It is easily seen (see, e.g., [2]) that the operator **Dom** is antimonotone, i.e., $\mathbf{R} \preceq \mathbf{S}$ implies $\mathbf{Dom}\,\mathbf{R} \succeq \mathbf{Dom}\,\mathbf{S}$. Our starting point in the proof of Theorems 1.1 and 1.2 is the following general result (see Appendix B for a proof):

Theorem 2.1 ([2, Lemma 5.1], [8, Theorem 5.5])**.** *For every n-tuple $\mathbf{P}$ of sites (in any metric space) there exist n-tuples $\mathbf{R}$ and $\mathbf{S}$ such that $\mathbf{R} = \mathbf{Dom}\,\mathbf{S}$ and $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$. Moreover, for every n-tuples $\mathbf{R}', \mathbf{S}'$ with $\mathbf{R}' = \mathbf{Dom}\,\mathbf{S}'$ and $\mathbf{S}' = \mathbf{Dom}\,\mathbf{R}'$ we have $\mathbf{R} \preceq \mathbf{R}', \mathbf{S}' \preceq \mathbf{S}$ (and in particular, $\mathbf{R} \preceq \mathbf{S}$).*

We finish this section with a simple geometric lemma. It was used, in a less general setting, in [2] (proof of Lemma 4.3).

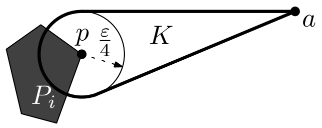

FIGURE 4. The cone K.

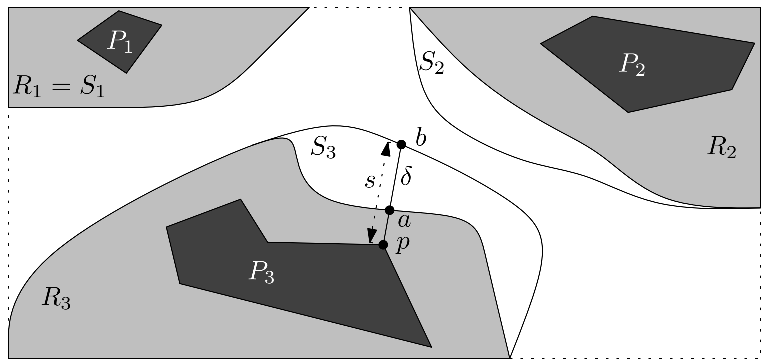

FIGURE 5. The setting of the proof of Theorem 1.1 (a schematic picture).

Observation 2.2. *Let* $\mathbf{P}$ *be an* n*-tuple of sites (in an arbitrary metric space), and suppose that* $\varepsilon := \min_{i\neq j} \mathrm{dist}(P_i, P_j) > 0$ *and that* $\mathbf{R}$ *and* $\mathbf{S}$ *satisfy* $\mathbf{R} = \mathbf{Dom}\,\mathbf{S}$ *and* $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$*. Then* $\mathrm{dist}(P_i, \bigcup_{j\neq i} S_j) \geq \frac{\varepsilon}{2}$*, and consequently, the* $\frac{\varepsilon}{4}$*-neighborhood of each* P_i *is contained in* R_i*.*

Proof. We recall the simple proof from [2]. We first note that $\mathbf{V} = (V_1, \ldots, V_n) := \mathbf{Dom}\,\mathbf{P}$ is the classical Voronoi diagram of $\mathbf{P}$, and the open $\frac{\varepsilon}{2}$-neighborhood of P_i does not intersect $\bigcup_{j\neq i} V_j$. Since $\mathbf{P} \preceq \mathbf{R}$, we have $\mathbf{Dom}\,\mathbf{P} \succeq \mathbf{Dom}\,\mathbf{R} = \mathbf{S}$, and hence the open $\frac{\varepsilon}{2}$-neighborhood of P_i is disjoint from $\bigcup_{j\neq i} S_j$ as well, as claimed. □

3. The Euclidean case

Here we prove Theorem 1.1; throughout this section, dist denotes the Euclidean distance. In addition to Theorem 2.1 and Observation 2.2, we also need the next lemma.

Lemma 3.1 (Cone lemma, Euclidean case)**.** *Let* $\mathbf{P}$ *be an* n*-tuple of (nonempty closed) sites in* $\mathbb{R}^d$ *with the Euclidean metric with* $\varepsilon := \min_{i\neq j} \mathrm{dist}(P_i, P_j) > 0$*, and let* $\mathbf{R}$ *and* $\mathbf{S}$ *satisfy* $\mathbf{R} = \mathbf{Dom}\,\mathbf{S}$ *and* $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$*. Let* a *be a point of some* R_i*, and let* $p \in P_i$ *be a point of the corresponding site closest to* a *(such a nearest point exists by compactness). Then the set*

$$K := \mathrm{conv}\big(\{a\} \cup \mathrm{B}(p, \tfrac{\varepsilon}{4})\big)$$

is contained in R_i*; see Fig. 4.*

The following proof is rather specific for the Euclidean metric (the lemma fails for the ℓ_1 metric, for example).

Proof. Both a and $\mathrm{B}(p, \frac{\varepsilon}{4})$ are contained in $\mathrm{dom}(p, \bigcup_{j\neq i} S_j)$ (the latter by Observation 2.2). For the Euclidean metric, the dominance region of a point over any set is convex, since it is the intersection of halfspaces. Hence $K \subseteq \mathrm{dom}(p, \bigcup_{j\neq i} S_j) \subseteq R_i$. □

Now we describe the general strategy of the proof of Theorem 1.1. With $\mathbf{R}$ and $\mathbf{S}$ as in Theorem 2.1, it suffices to prove $\mathbf{R} = \mathbf{S}$. For contradiction, we assume that it is not the case, i.e., that $R := \bigcup_{i=1}^n R_i$ is properly contained in $S := \bigcup_{i=1}^n S_i$; see the schematic illustration in Fig. 5.

For a point $b \in S \setminus P$, let $s(b) := \mathrm{dist}(b, P)$ be the distance from the nearest site, and let $p = p(b) \in P_i$ be a point where this distance is attained. Let $a = a(b)$ be the closest point to b that lies in the intersection of R_i with the segment bp. It is easily seen, using the triangle inequality, that p is also a nearest point of P to a. Thus, the set K in Lemma 3.1 is

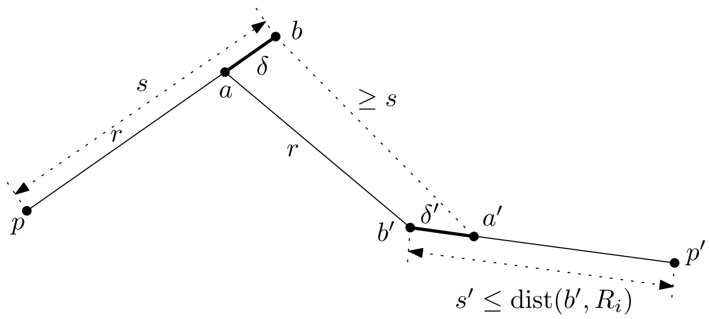

FIGURE 6. The construction of b'.

contained in R_i, and in particular, a is the only intersection of the segment bp with ∂R_i. We set $\delta(b) := \mathrm{dist}(b, a)$. The parameters $s(b)$ and $\delta(b)$ will measure, in some sense, how much S differs from R "at b".

Assuming $\mathbf{R} \neq \mathbf{S}$, we choose a point $b_0 \in S \setminus R$. Then, using b_0, we find $b_1 \in S \setminus R$ where S differs from R "more than" at b_1. Iterating the same procedure we obtain an infinite sequence $b_0, b_1, b_2, b_3, \ldots$ of points, and the difference will "grow" beyond bounds, while, on the other hand, it has to stay bounded—and this way we reach a contradiction.

More concretely, for every integer $t \geq 1$ we will construct b_t from b_{t-1} so that, with $s := s(b_{t-1})$, $s' := s(b_t)$, $\delta := \delta(b_{t-1})$, and $\delta' := \delta(b_t)$, we have

(A) $s' \leq s - \alpha$, or

(B) $s' \leq s - \delta$ and $\delta' \geq \delta$,

where $\alpha > 0$ is a constant that depends on $s_0 := s(b_0)$ and ε, but not on t.

Thus, as t increases, $s(b_t)$ keeps decreasing. Since $s(b_t)$ is bounded from below by $\frac{\varepsilon}{4}$ by Observation 2.2, case (A) can happen only finitely many times. Therefore, from some t on, we have case (B) only. But this also causes $s(b_t)$ to decrease towards 0—a contradiction.

It remains to describe the construction of b_t from b_{t-1}, and this is done in the next lemma.

Lemma 3.2. *For every s_0 and $\varepsilon > 0$ there exists $\alpha > 0$ such that if $b \in S \setminus R$ satisfies $s := s(b) \leq s_0$, then there exists another point $b' \in S \setminus R$ such that $s' := s(b')$, $\delta := \delta(b)$ and $\delta' := \delta(b')$ satisfy* (A) *or* (B).

Proof. Let $b \in S_i$, let $a := a(b)$, $p := p(b)$, and write $r = \mathrm{dist}(a, p)$; see Fig. 6. Since $a \in \partial R_i$ and $\mathbf{R} = \mathbf{Dom}\,\mathbf{S}$, there exist $j \neq i$ and $b' \in S_j$ with $\mathrm{dist}(a, b') = r$. If there are several possible b', we choose one of them arbitrarily.

First we check that $b' \notin R$, or in other words, that $\delta' > 0$. During this step we also derive a lower bound for δ' that will be useful later. Since $b \in S$, $a' \in R$, and $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$, we have $\mathrm{dist}(a', b) \geq s$. Then we bound, using the triangle inequality,

$$\delta' \geq \mathrm{dist}(a', b) - \mathrm{dist}(b, b') \geq s - \mathrm{dist}(b, b'). \tag{1}$$

Supposing for contradiction that $\delta' = 0$, we get $\mathrm{dist}(b, b') = s$. But the triangle inequality gives $\mathrm{dist}(b, b') \leq \mathrm{dist}(b, a) + \mathrm{dist}(a, b') = r + \delta = s$, and hence the triangle inequality here holds with equality. For the Euclidean metric, this can happen only if a lies on the segment bb', and then b' has to coincide with p, which is impossible. So $\delta' > 0$ indeed.

Next, since $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$ and $b' \in S$, we have $s' \leq \mathrm{dist}(b', R_i)$. An obvious upper bound for $\mathrm{dist}(b', R_i)$ is $\mathrm{dist}(b', a) = r = s - \delta$, and thus the first inequality in (B), namely, $s' \leq s - \delta$, always holds.

Moreover, if $\delta \geq \alpha$, then $s' \leq s - \delta \leq s - \alpha$, and we have (A). For the rest of the proof we thus assume that $\delta < \alpha$ (where α hasn't been fixed yet—so far we're free to choose it as a positive function of ε and s_0 in any way we like).

Let us consider the ball $\mathrm{B}(b', r)$; see Fig. 7. If it contains b, as in the left picture, we have $\mathrm{dist}(b', b) \leq r$, and thus by (1) we have $\delta' \geq s - r = \delta$. Then (B) holds. Thus, the last case to deal with is $b \notin \mathrm{B}(b', r)$.

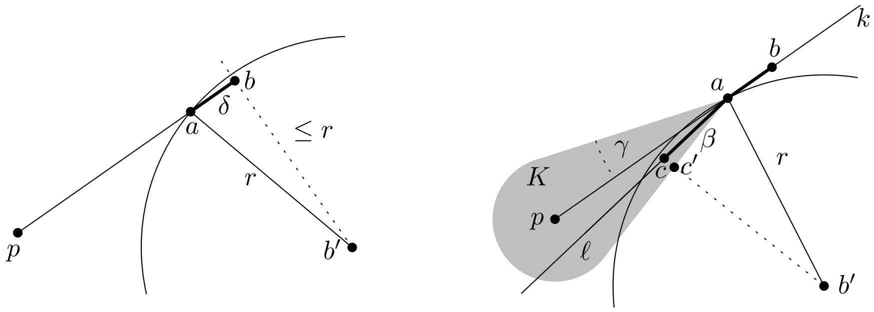

FIGURE 7. The r-ball around b'.

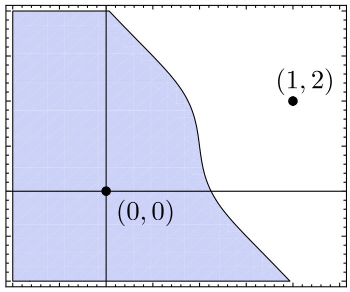

FIGURE 8. The dominance region of the point $(0,0)$ against $(2,1)$ in the ℓ_4 norm.

Let us consider the cone $K = \mathrm{conv}(\{a\} \cup \mathrm{B}(p, \frac{\varepsilon}{4}))$ as in Lemma 3.1. Its opening angle γ is bounded away from 0 in terms of ε and s_0.

Let Π be a 2-dimensional plane containing p, a, b'; it also contains b since p, a, b are collinear. Let k be the ray originating at a and containing b, and let ℓ be the ray in Π originating at a and making the angle $\pi - \frac{\gamma}{2}$ with k (on the side of b'); see Fig. 7 right.

Since the angle of the rays k and ℓ is bounded away from the straight angle, the Euclidean ball $\mathrm{B}(b', r)$ cuts a segment of significant length β from at least one of these rays; here β can be bounded from below by a positive quantity depending only on s_0 and ε. So far we haven't fixed α, and so now we can make sure that $\alpha < \beta$. Since we assume $b \notin \mathrm{B}(b', r)$, the segment of length β cut out by $\mathrm{B}(b', r)$ can't belong to the ray k. So the situation is as in Fig. 7 right: $\mathrm{B}(b', r)$ contains the initial segment ac of ℓ of length β. Hence $\mathrm{dist}(b', c) \leq r$.

The distance $\mathrm{dist}(c, \mathbb{R}^d \setminus K)$ is bounded away from 0 in terms of β and γ, and so we may fix α so that $\mathrm{dist}(c, \mathbb{R}^d \setminus K) \geq \alpha$.

Let c' be the point where the segment $b'c$ meets the boundary of K. We have

$$\mathrm{dist}(b', K) \leq \mathrm{dist}(b', c') = \mathrm{dist}(b', c) - \mathrm{dist}(c, c') \leq r - \mathrm{dist}(c, \mathbb{R}^d \setminus K) \leq r - \alpha.$$

Then, finally, using $K \subseteq R_i$, we have

$$s' \leq \mathrm{dist}(b', R_i) \leq \mathrm{dist}(b', K) \leq r - \alpha < s - \alpha,$$

and so (A) holds. This concludes the proof of Lemma 3.2, as well as that of Theorem 1.1. $\square$

4. The case of smooth and rotund norms

In this section we establish Theorem 1.2. We begin with the part where the proof differs from the Euclidean case: the cone lemma. In the Euclidean case, we used the fact that for points $p \neq q$, $\mathrm{dom}(p, q)$ is a halfspace, and consequently, $\mathrm{dom}(p, X)$ is convex for arbitrary X. For other norms $\mathrm{dom}(p, q)$ need not be convex, though; see Fig. 8.

We have at least the following convexity result.

Lemma 4.1. *Let us consider $\mathbb{R}^d$ with an arbitrary norm $\|\cdot\|$, let H be a closed halfspace, and let $p \notin H$ be a point. Then $\mathrm{dom}(p, H)$ is convex.*

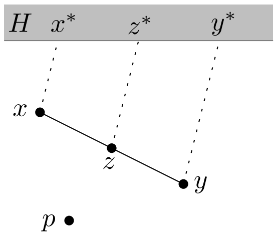

FIGURE 9. The dominance region of a point against a halfspace.

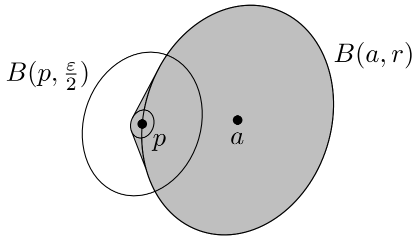

FIGURE 10. The sets C (shaded) and D.

Consequently, if the complement of a closed set $A \subseteq \mathbb{R}^d$ is convex and $p \notin A$, then $\mathrm{dom}(p, A)$ *is convex.*

Proof. Let $x \notin H$ be a point and let $x^* \in \partial H$ be a point where $\mathrm{dist}(x, H)$, the distance of x to H measured by $\|\cdot\|$, is attained. If $y \notin H$ is another point and $y^* \in \partial H$ is the point such that the vectors $x - x^*$ and $y - y^*$ are parallel, then $\|y - y^*\| = \mathrm{dist}(y, H)$; see Fig. 9.

Now let $x, y \in \mathrm{dom}(p, H)$, let x^*, y^* be as above, set $z := (x + y)/2$, and let z^* be defined analogously to y^*. Then we get $\mathrm{dist}(z, H) = \|z - z^*\| = (\|x - x^*\| + \|y - y^*\|)/2 = (\mathrm{dist}(x, H) + \mathrm{dist}(y, H))/2$. From this $z \in \mathrm{dom}(p, H)$ is immediate, since $\|p - z\| \le (\|p - x\| + \|p - y\|)/2 \le (\mathrm{dist}(x, H) + \mathrm{dist}(y, H))/2 = \mathrm{dist}(z, H)$. This proves the first part of the lemma.

The second part follows easily: A can be expressed as a union of closed halfspaces H, and $\mathrm{dom}(p, A)$ is the intersection of the convex sets $\mathrm{dom}(p, H)$. □

Now we prove a cone lemma, similar to Lemma 3.1:

Lemma 4.2 (Cone lemma for rotund norms)**.** *Let $\|\cdot\|$ be a rotund norm on $\mathbb{R}^d$. Suppose that an n-tuple $\mathbf{P}$ of sites satisfies $\varepsilon := \min_{i \ne j} \mathrm{dist}(P_i, P_j) > 0$, and $\mathbf{R}$ and $\mathbf{S}$ satisfy $\mathbf{R} = \mathbf{Dom}\,\mathbf{S}$ and $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$. Then for every $s_0 > 0$ there is $\rho > 0$ (also depending on ε and on $\|\cdot\|$) such that the following holds: If $a \in R_i$ with $r := \mathrm{dist}(a, P_i) \le s_0$ and $p \in P_i$ is a point attaining the distance $\mathrm{dist}(a, P_i)$, then the set*

$$K := \mathrm{conv}\big(\{a\} \cup \mathrm{B}(p, \rho)\big)$$

is contained in R_i.

Proof. As in the Euclidean case, we begin by observing that $a \in \mathrm{dom}(p, \bigcup_{j \ne i} S_j)$ and also $\mathrm{B}(p, \frac{\varepsilon}{4}) \subseteq \mathrm{dom}(p, \bigcup_{j \ne i} S_j)$ by Observation 2.2. Thus, the set $D := \mathrm{B}(a, r) \cup \mathrm{B}(p, \frac{\varepsilon}{2})$ is contained in the closure of $\mathbb{R}^d \setminus \bigcup_{j \ne i} S_j$. We now want to find a open convex subset $C \subseteq D$ such that a and $B(p, \rho)$ are contained in $\mathrm{dom}(p, \mathbb{R}^d \setminus C)$, since the latter region is convex by Lemma 4.1 and thus it contains K as well.

We let C be the interior of $\mathrm{conv}(B(a, r) \cup B(p, 2\rho))$ with ρ sufficiently small (the restrictions on it will be apparent from the proof below); see Fig. 10. It is clear that $\{a\} \cup B(p, \rho) \subseteq \mathrm{dom}(p, \mathbb{R}^d \setminus C)$, and so it remains to prove $C \subseteq D$.

To this end, it is sufficient to prove the following: If $B := \mathrm{B}(0, 1)$ is the unit ball of $\|\cdot\|$ and $\eta > 0$ is given, then there exists $\delta > 0$ such that for every $x \in \mathbb{R}^d$ with $\|x\| \le 1 + \delta$, the "cap"

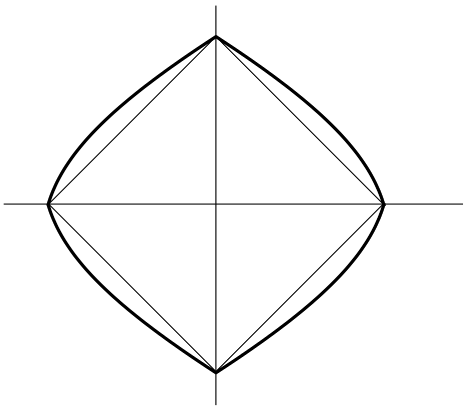

FIGURE 11. A schematic illustration of the unit ball of $\|\cdot\|_{(1)}$.

$\operatorname{conv}(B \cup \{x\}) \setminus B$ has diameter at most η. This is a well-known and easily proved property of uniformly convex norms. (Proof sketch: If x with $\|x\| = 1+\delta$ has a cap of large diameter, then there is z of norm 1 and half of the diameter away from x such that the line xz avoids the interior of B. Let y be the other intersection of this line with $\partial \mathrm{B}(0, 1+\delta)$—then xy is a long segment that cuts in $\mathrm{B}(0, 1+\delta)$ into depth only δ.) □

Proof of Theorem 1.2. The overall strategy of the proof is exactly as for Theorem 1.1 (see Section 3). The constant α in (A) may also depend on the considered norm $\|\cdot\|$. This quantification also needs to be added in the appropriate version of Lemma 3.2.

In the proof of that lemma, the first place where we use a property not shared by all norms is below (1); we need that the triangle inequality may hold with equality only for collinear points—this remains true for all *rotund* norms.

Then we proceed as in the Euclidean case, introducing the the cone $K = \operatorname{conv}(\{a\} \cup B(p, \rho))$ as in Lemma 4.2. There is some $\gamma > 0$, depending on ε, s_0, and the norm $\|\cdot\|$, such that the appropriate Euclidean cone with opening angle γ is contained in K. (Here and in the sequel we implicitly use the fact that every norm on $\mathbb{R}^d$ is between two constant multiples of the Euclidean norm, which is well known and immediate by compactness.)

We define the rays k and ℓ, again following the Euclidean proof. For the next step, we need that, since the angle of these rays is bounded away from the straight angle, at least one of k, ℓ cuts a segment of a significant length β from the ball $\mathrm{B}(b', r)$. It is easy to see that this property follows from the *smoothness* of the norm. The rest of the proof goes through unchanged. □

5. Non-uniqueness examples

As we saw in the introduction, two point sites with the same x-coordinate have at least two zone diagrams under the ℓ_1 metric. Here we show that only the non-smoothness (sharp corners) of the ℓ_1 unit ball is essential for this example, while the straight edges can be replaced by curved ones.

Proposition 5.1. *There exists a rotund norm in the plane, arbitrarily close to the ℓ_1 norm, such that two distinct point sites with the same x-coordinate have (at least) two different zone diagrams.*

The appropriate norm is not difficult to describe, but proving the non-uniqueness of the zone diagram is more demanding, since it seems hard to find an explicit description of a zone diagram for non-polygonal norms.

Informally, we construct the desired norm by slightly "inflating" the unit ball of the planar ℓ_1 norm, so that the edges bulge out and the norm becomes rotund. It is important that the inflation is asymmetric, as is schematically indicated in Fig. 11 (in the "real" example we inflate much less). We will denote the resulting norm by $\|\cdot\|_{(1)}$; the subscript should remind of "inflated ℓ_1" graphically.

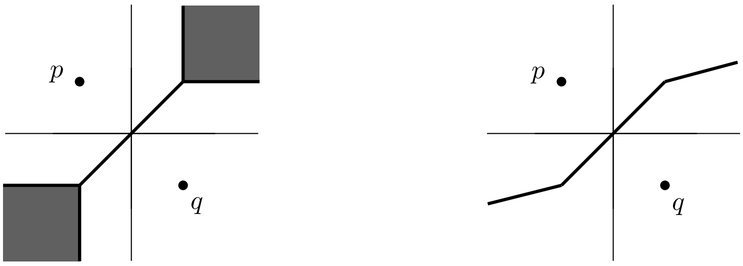

FIGURE 12. The bisector of p and q under the ℓ_1 norm and under $\|\cdot\|_{(1)}$ (schematic).

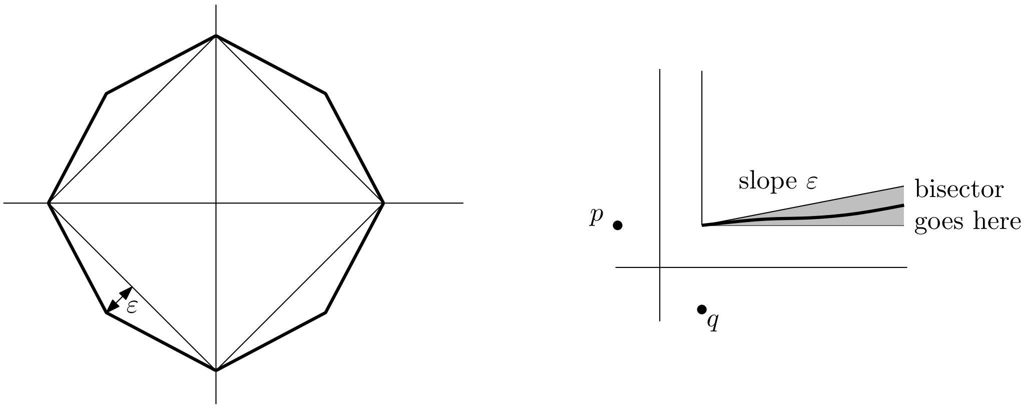

FIGURE 13. The conditions in Lemma 5.2.

To explain the purpose of the asymmetry in our example, we consider the bisector of the points $p = (-1, 1)$ and $q = (1, -1)$, i.e., the set of all points equidistant to p and q. For the ℓ_1 norm, the bisector is "fat", as shown in Fig. 12 left—it consists of a segment and two quadrants. By a small inflation, which makes the norm rotund, the middle segment of the bisector is changed only very slightly, but the "ambiguity" of the ℓ_1 bisector in the quadrants is "resolved", and the quadrants collapse to (possibly curved) rays. Now if the inflation were symmetric, we would get straight rays with slope 1 in the bisector, but with an asymmetric inflation, we can get a (positive) slope as small as we wish.

In order to establish the required properties of the bisector formally, a safe route (if perhaps not the most conceptual one) is to describe $\|\cdot\|_{(1)}$ analytically. The rays of the bisector will be slightly curved rather than straight, but for the zone diagram construction this will do as well.

Lemma 5.2. *For every $\varepsilon > 0$ there exists a rotund norm $\|\cdot\|_{(1)}$ in the plane, whose unit ball contains the ℓ_1 unit ball and is contained in the octagon as in Fig. 13 left, such that the portion of the bisector of the points $p = (-1, 1)$ and $q = (1, -1)$ lying in the quadrant $\{\,(x, y) : x, y \geq 1\,\}$ is an x-monotone curve lying below the line $y = \varepsilon(x - 1) + 1$ (Fig. 13 right).*

See Appendix C for a proof.

Proof of Proposition 5.1. We show that the zone diagram of the sites $p^- = (0, -1)$ and $p^+ = (0, +1)$ under the norm $\|\cdot\|_{(1)}$ as in the lemma, with ε sufficiently small, is not unique.

First we consider the zone diagram only inside the vertical strip

$$V := \{\,(x, y) \in \mathbb{R}^2 : x \in [-2, 2]\,\}.$$

Let R_0^+ be the region as in Fig. 14, i.e., the part of the region of p^- within V in an ℓ_1 zone diagram of p^-, p^+. Let S_0^+ be obtained by pulling the bottom vertex of R_0^+ downward by η (which is another small positive parameter), and let R_0^-, S_0^- be the reflections of R_0^+, S_0^+ by the x-axis.

Let us consider the region $\mathrm{dom}(p^-, R_0^+)$ inside V (distances measured by our norm $\|\cdot\|_{(1)}$). For every point $x \in V$ below R_0^+, the $\|\cdot\|_{(1)}$-distance to R_0^+ coincides with the ℓ_1 distance,

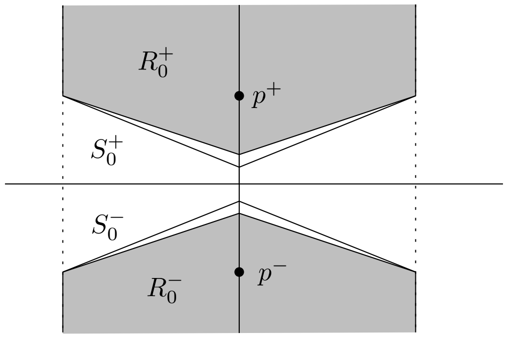

FIGURE 14. The regions $R_0^+, S_0^+, R_0^-, S_0^-$ in the vertical strip V.

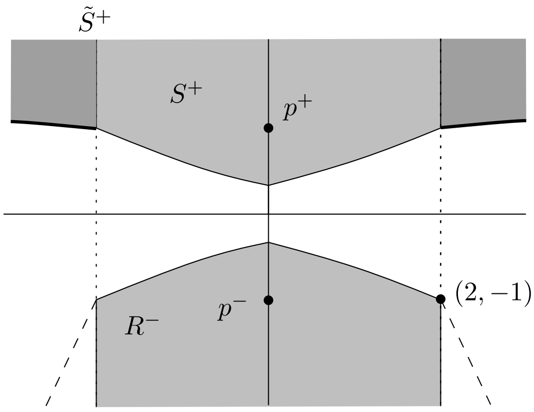

FIGURE 15. The region $\tilde{S}^+$ defined using bisectors, and a region containing $\tilde{R}^-$.

which is simply the length of the vertical segment from x to ∂R_0^+. From this it is clear that $\mathrm{dom}(p^-, R_0^+) \supseteq R_0^-$ (since R_0^- is the dominance region of p^- against R_0^+ in the ℓ_1 metric, and $\|\cdot\|_{(1)} \le \|\cdot\|_1$). Moreover, it's easy to check that for ε (the parameter controlling the choice of $\|\cdot\|_{(1)}$) sufficiently small, we also have $\mathrm{dom}(p^-, R_0^+) \subseteq S_0^-$.

Thus, we have $R_0^- \subseteq \mathrm{dom}(p^-, R_0^+) \subseteq S_0^-$, and by the vertical symmetry we also get $R_0^+ \subseteq \mathrm{dom}(p^+, R_0^-) \subseteq S_0^+$. Arguing as in either of the proofs of Theorem 2.1, we get that there exist regions R^+, R^-, S^+, S^-, where R^- is the reflection of R^+, S^- is the reflection of S^+, such that $R_0^- \subseteq R^+ \subseteq S^+ \subseteq S^+$, and (R^-, S^+) is a zone diagram of (p^-, p^+) (and so is (S^-, R^+), but we actually have $R^+ = S^+$, although we will neither need this nor prove it).

All of this refers to the vertical strip V (so, formally, the metric space in these arguments is V with the $\|\cdot\|_{(1)}$ metric). Now we move on to the full plane $\mathbb{R}^2$, and we let $\tilde{S}^+$ be the region consisting of S^+ plus two parts of the upper halfplane outside V as in Fig. 15: The right part is delimited by a part of the bisector of p^+ and $(2,-1)$ (drawn thick), and the left part by a part of the bisector of p^+ and $(-2,-1)$.

Now we set $\tilde{R}^- := \mathrm{dom}(p^-, \tilde{S}^+)$. The distance of points inside $V \setminus S^+$ to $\tilde{S}^+$ is still the vertical distance, i.e., the same as the distance to S^+, and so $\tilde{R}^- \cap V = R^-$. For the part of $\tilde{R}^-$ outside V, we don't need an exact description—it is sufficient that it lies below the dashed rays in Fig. 15 (using the property of the bisectors as in Lemma 5.2, one can see that these rays can be taken as steep as desired, by setting ε sufficiently small). From this we can see that for every point of the upper halfplane on the right of V, the nearest point of $\tilde{R}^-$ is the corner $(2,-1)$.

Therefore, $\mathrm{dom}(p^+, \tilde{R}^-) = \tilde{S}^+$, and hence $(\tilde{R}^-, \tilde{S}^+)$ is a zone diagram of (p^-, p^+). But the mirror reflection of this zone diagram about the x-axis yields another, different zone diagram. $\square$

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Appendix A. Proof of Theorem 1.3

Proposition 5.1 showed that the assumption of smoothness in Theorem 1.2 cannot be dropped, even for the simplest case of two singleton sites in the plane. Theorem 1.3, which we will prove here, states that the rotundity assumption can be dropped in this special case.

Smoothness of the norm means that a metric ball has a unique supporting halfspace at every point in its surface. Thus, for a nonzero vector a, we can define $\top_a^{>0}$ to be the open halfspace that touches (but not intersects) the ball $\mathrm{B}(-a, \|a\|)$ at the origin. We write $\top_a^{\leq 0} = \mathbb{R}^d \setminus \top_a^{>0}$ and $\top_a^{\geq 0} = \top_{-a}^{\leq 0}$. For nonzero vectors a and b, define $a \sim b$ when $\top_a^{>0} = \top_b^{>0}$. Then $\sim$ is an equivalence relation. It is easy to see (Fig. 16) that for nonzero vectors $a_1, \ldots, a_m$, we have

$$\|a_1 + \cdots + a_m\| = \|a_1\| + \cdots + \|a_m\| \qquad \text{if and only if} \qquad a_1 \sim \cdots \sim a_m. \tag{2}$$

Lemma A.1. *Let $\|\cdot\|$ be a smooth norm on $\mathbb{R}^d$. Then there are positive numbers α and β such that for any unit vectors u, v with $\|u + v\| > 2 - \beta$, we have $\|u - \alpha v\| \leq 1$.*

Proof. The angle σ_u between a unit vector u and $\top_u^{\leq 0}$ is a continuous function of u, and hence attains a positive minimum σ. Let $\top_u^{\geq \sigma/2}$ (and $\top_u^{\leq \sigma/2}$) be the set of vectors (including 0) that make an angle $\geq \sigma/2$ (and $\leq \sigma/2$) with $\top_u^{\leq 0}$ (Fig. 17). We find the desired α and β as follows.

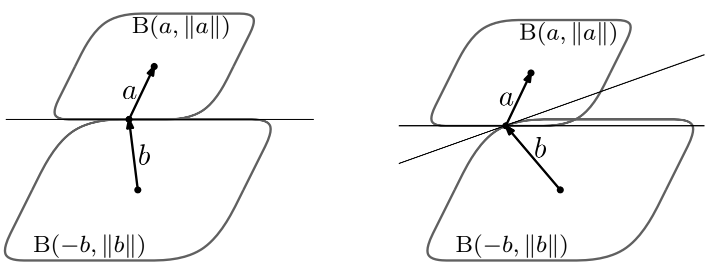

Figure 16. $\|a + b\| = \|a\| + \|b\|$ if and only if $a \sim b$ (equation (2) with $m = 2$).

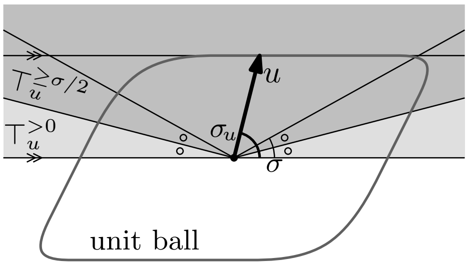

FIGURE 17. $\top_u^{\geq\sigma/2}$ is the set of vectors that are significantly closer to u than to $-u$.

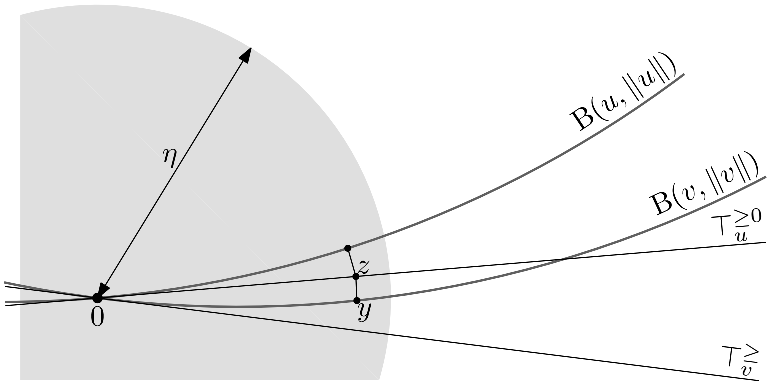

FIGURE 18. When u and v are close, $y \in \mathrm{B}(v,\|v\|)$ is not very far from $\mathrm{B}(u,\|u\|)$.

For unit vectors u and v with $v \in \top_u^{\geq\sigma/2}$, let $\alpha_{u,v}$ be the length of the segment that the unit ball cuts out from the line $u+\mathbb{R}v$. In other words, $\alpha_{u,v}$ is the unique positive number such that $\|u-\alpha_{u,v}v\| = 1$. Then $\alpha_{u,v}$ is continuous in u and v, and thus attains a positive minimum α.

For unit vectors u and v with $v \in \top_u^{\leq\sigma/2}$, let $\beta_{u,v} = 2 - \|u+v\|$. Then $\beta_{u,v}$ is positive and continuous in u and v, and thus attains a positive minimum β.

Since $\top^{\geq\sigma/2}$ and $\top^{\leq\sigma/2}$ covers the whole space, α and β have the stated property. □

Lemma A.2. *Let* $\|\cdot\|$ *be a smooth norm on* $\mathbb{R}^d$*. For any* $\kappa > 0$*, there is* $\varepsilon > 0$ *such that, for any vectors* u*,* v *with* $\|u\|, \|v\| \geq 1$ *and* $\|u-v\| < \varepsilon$*, we have* $\mathrm{dist}(y, \mathrm{B}(u,\|u\|)) < \kappa\|y\|$ *for any* $y \in \mathrm{B}(v,\|v\|)$*.*

Proof. Since $\mathrm{dist}(y, \mathrm{B}(u,\|u\|)) \leq 2\varepsilon$, it is clear that, for any constant $\eta > 0$, the claim holds if we consider only those y with $\|y\| \geq \eta$. Therefore, it suffices to prove the existence of $\eta > 0$, depending on $\|\cdot\|$ and κ, such that the claim holds for any y with $\|y\| < \eta$.

We find the desired η and ε as follows (Fig. 18). Since the norm is smooth, the surface of a ball looks like a hyperplane locally at each point. Thus, there exists $\eta > 0$ such that for any $u \in \mathbb{R}^d$ with $\|u\| \geq 1$ and any $z \in \top_u^{\geq 0}$ with $\|z\| < \eta(1+\kappa/2)$, we have $\mathrm{dist}(z, \mathrm{B}(u,\|u\|)) \leq \kappa\|z\|/(2+\kappa)$. Also, since changing slightly a vector u of length 1 or greater moves $\top^{\geq 0}$ only slightly, there is $\varepsilon > 0$ so small that for any vectors u, v of length 1 or greater with $\|u-v\| < \varepsilon$, we have $\mathrm{dist}(y, \top_u^{\geq 0}) \leq \kappa\|y\|/(2\eta)$ for all $y \in \top_v^{\geq 0}$.

Since $y \in \mathrm{B}(v,\|v\|) \subseteq \top_v^{\geq 0}$, we have $\mathrm{dist}(y, \top_u^{\geq 0}) \leq \kappa\|y\|/2$ by our choice of ε. Let $z \in \top_u^{\geq 0}$ be a point attaining this distance. Since $\|z\| \leq \|y\| + \|z-y\| \leq \|y\| + \kappa\|y\|/2 = \|y\|(1+\kappa/2) \leq \eta(1+\kappa/2)$, we have $\mathrm{dist}(z, \mathrm{B}(u,\|u\|)) \leq \kappa\|z\|/(2+\kappa) \leq \kappa\|y\|/2$ by our choice of η. These imply $\mathrm{dist}(y, \mathrm{B}(u,\|u\|)) < \kappa\|y\|$ by the triangle inequality. □

Lemma A.3. *Let* $\|\cdot\|$ *be a smooth norm on* $\mathbb{R}^2$*. For unit vectors* u *and* v *with* $\|u-v\| < 2$*, there is* $\kappa > 0$ *such that for all* $y \in \mathrm{dom}(v,u) \setminus \mathrm{B}(v,1)$ *sufficiently close to the origin (Fig. 19),* $\mathrm{dist}(y, \mathrm{B}(u,1)) \geq \kappa\|y\|$*.*

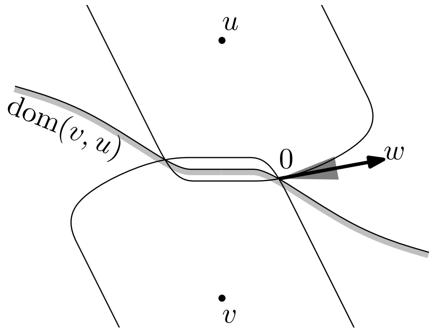

FIGURE 19. The conclusion of Lemma A.3 states that $\mathrm{dom}(v, u)$ and the boundary of $\mathrm{B}(u, 1)$ "make a positive angle" at the origin. We prove this by showing that there is a cone (shaded) whose axis is the tangent vector w and which does not overlap $\mathrm{dom}(v, u)$.

Proof. Because $\|u - v\| < 2$, the vectors u and $-v$ do not share the tangent. Therefore, there is a (unique) unit vector $w \in \top_u^{\geq 0} \cap \top_u^{\leq 0}$ that heads out of $\mathrm{B}(v, 1)$. Since

$$\lim_{\delta \to 0} \frac{\|u - \delta w\| - 1}{\delta} = 0, \qquad \beta := \lim_{\delta \to 0} \frac{\|v - \delta w\| - 1}{\delta} > 0,$$

there exists $\delta_0 > 0$ so small that for all positive $\delta < \delta_0$, we have

$$\frac{\|u - \delta w\| - 1}{\delta} < \frac{1}{3}\beta, \qquad \frac{\|v - \delta w\| - 1}{\delta} > \frac{2}{3}\beta,$$

and hence $\|u - \delta w\| < \|v - \delta w\| - \beta\delta/3$. This implies that $\|u - x\| < \|v - x\|$ for all $x \in \mathrm{B}(\delta w, \beta\delta/6)$. Thus, $\mathrm{dom}(v, u)$ is disjoint from a cone (except at the origin) whose vertex is at the origin and axis is the vector w (see Fig. 19). This implies what is stated. $\square$

Now we look at the situation of Theorem 1.3. Let $\mathbf{R} = (R_0, R_1)$ and $\mathbf{S} = (S_0, S_1)$ be pairs satisfying $\mathbf{R} \preceq \mathbf{S}$ and $\mathbf{R} = \mathbf{Dom}\,\mathbf{S}$, $\mathbf{S} = \mathbf{Dom}\,\mathbf{R}$ (which exist by Theorem 2.1). As before, it suffices to show that $\mathbf{R} = \mathbf{S}$. Suppose otherwise. Then $h = \min\{\mathrm{dist}(p_0, S_0 \setminus R_0), \mathrm{dist}(p_1, S_1 \setminus R_1)\}$ exists.

Lemma A.4. *In the above setting, if a point $c \in \overline{S_0 \setminus R_0}$ satisfies $\|c - p_0\| = h$, then*
(a) $\|c - p_1\| = 2h$;
(b) *there is a point $c' \in \overline{S_1 \setminus R_1}$ satisfying $\|c' - c\| = \|c' - p_1\| = h$.*

Proof. Note that $c \in R_0$, since otherwise $S_0 \setminus R_0$ intersects a part of the segment cp_0 of positive length, contradicting the minimality of h.

There is a sequence $(x_i)_{i \in \mathbb{N}}$ of points in $S_0 \setminus R_0$ that converges to c. For each $i \in \mathbb{N}$, let $y_i \in S_1$ be a closest point to x_i. Since $x_i \in S_0 \setminus R_0$, we have $\|y_i - x_i\| = \mathrm{dist}(x_i, S_1) < \|p_0 - x_i\|$ and $y_i \in S_1 \setminus R_1$. The sequence $(y_i)_{i \in \mathbb{N}}$ has a subsequence $(y_{j_i})_{i \in \mathbb{N}}$ that converges to a point $c' \in \overline{S_1 \setminus R_1}$ (Fig. 20). Note that

$$\|c' - p_1\| \leq \|c - c'\| = \lim_{i \to \infty} \|x_{j_i} - y_{j_i}\| \leq \lim_{i \to \infty} \|p_0 - x_{j_i}\| = \|p_0 - c\| = h,$$

where the first inequality is by $c' \in S_1$ and $c \in R_0$. In fact, this holds in equality by the minimality of h. We have proved (b).

For each i, since $S_1 \setminus R_1$ intersects a part of the segment $y_{j_i}c'$ of positive length, $y_{j_i} \notin \mathrm{B}(p_1, h)$ by the minimality of h. Also, $y_{j_i} \in S_1 \subseteq \mathrm{dom}(p_1, c)$. As i increases, y_{j_i} comes arbitrarily close to c'. Hence, if (a) is not true, Lemma A.3 gives a constant $\kappa > 0$ such that $\mathrm{dist}(y_{j_i}, \mathrm{B}(c, h)) \geq \kappa\|y_{j_i} - c'\|$ for all but finitely many i. On the other hand, since y_{j_i} is in $\mathrm{B}(x_{j_i}, \|x_{j_i} - c'\|)$ and $(x_{j_i})_{i \in \mathbb{N}}$ converges to c, Lemma A.2 shows that $\mathrm{dist}(y_{j_i}, \mathrm{B}(c, h)) < \kappa\|y_{j_i} - c'\|$ for all but finitely many i. This is a contradiction. We have proved (a). $\square$

Lemma A.5. *In the above setting, $\|p_0 - p_1\| = 3h$.*

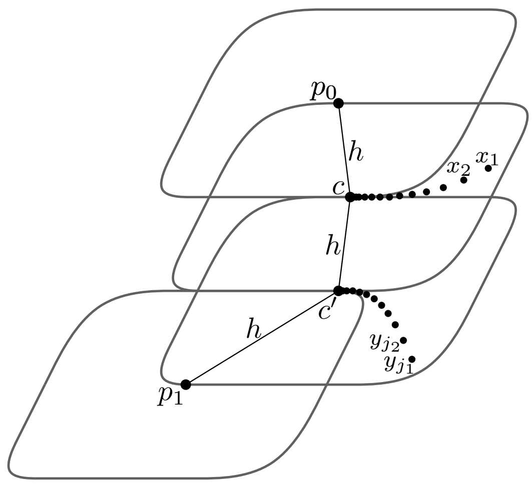

FIGURE 20. Lemma A.4.

Proof. By the definition of h, there is a point $c \in \overline{S_0 \setminus R_0}$ satisfying $\|c - p_0\| = h$. By Lemma A.4(b), there is a point $c' \in \overline{S_1 \setminus R_1}$ satisfying $\|c' - c\| = \|c' - p_1\| = h$. By Lemma A.4(a) (and the same lemma with the sites swapped), $\|c - p_1\| = \|c' - p_0\| = 2h$. This implies $(c - p_0) \sim (c' - c) \sim (p_1 - c')$ and thus $\|p_0 - p_1\| = 3h$ by (2) at the beginning of this section. $\square$

To prove Theorem 1.3, we will construct a sequence $(b_t)_{t\in\mathbb{N}}$ of points in $R \setminus S$, as we did in Section 3. Recall that for each $i \in \{0, 1\}$ and $b \in S_i$, we define $a(b)$ to be the closest point to b that is in the intersection of R_i with the segment bp_i (note that since we do not have the cone lemma this time, the intersection of bp_i and ∂R_i is not always unique). As before, let $s(b) = \|b - p_i\|$ and $\delta(b) = \|b - a(b)\|$.

The proof goes as follows. This time, we begin with a point $b_0 \in S_0 \setminus R_0$ that is within distance $h + \varepsilon$ from the nearest site, for some small $\varepsilon > 0$ (such b_0 exists by the definition of h), and take $b_1, b_2, \ldots$ as we did in Section 3 using Lemma 3.2: For each $b_t \in S_i \setminus R_i$, we let $b_{t+1} \in S_{1-i} \setminus R_{1-i}$ be a point that is at the same distance from $a(b_t)$ as p_i is. Then each b_t will be also within distance $h + \varepsilon$ from the nearest site p_i. Because we have proved that the sites are $3h$ apart, and the path p_i-$a(b_t)$-b_{t+1}-p_{1-i} consists of three segments shorter than $h + \varepsilon$, the path must be "almost straight". This implies that we will always have the case (B) in Section 3 (Fig. 7 left):

Lemma A.6. *In the above setting, the following holds for some $\varepsilon > 0$: For each $i \in \{0, 1\}$ and $b \in S_i \setminus R_i$ satisfying $s := s(b) < h + \varepsilon$, there is $b' \in S_{1-i} \setminus R_{1-i}$ such that $\delta := \delta(b)$, $s' := s(b')$, $\delta' := \delta(b')$ satisfy* (B) *of Section 3 (i.e., $\delta' \geq \delta$ and $s' \leq s - \delta$).*

Proof. Let $\varepsilon := \min\{h\alpha, h\beta/3\}$, where α and β are as in Lemma A.1. Let b be as assumed. By the definition of $a := a(b)$, there is $b' \in S_{1-i}$ with $\|b' - a\| = \|a - p_i\|$. We show that this b' qualifies. Since $s' = \|b' - p_{1-i}\| \leq \|b' - a\| = \|a - p_{1-i}\| = s - \delta$, it suffices to prove that $\delta' \geq \delta$ (which would then imply $b' \notin R_{1-i}$).

By Lemma A.5, we have

$$\begin{aligned}\|b' - p_i\| &\geq \|p_{1-i} - p_i\| - \|p_{1-i} - b'\| = 3h - s' > 3h - s \geq 3h - (h + \varepsilon)\\ &= 2(h + \varepsilon) - 3\varepsilon \geq 2(h + \varepsilon) - \beta h > (h + \varepsilon)(2 - \beta) > \|a - p_i\|(2 - \beta).\end{aligned}$$

By this and $\|b' - a\| = \|a - p_i\|$, Lemma A.1 yields $\|(b' - a) - \alpha(a - p_i)\| \leq \|a - p_i\|$. This remains true if we decrease α, since $\mathrm{B}(0, \|a - p_i\|)$ is convex. So we replace α by $\|b - a\|/\|a - p_i\| \leq \varepsilon/h \leq \alpha$, obtaining $\|b' - b\| = \|(b' - a) - (b - a)\| \leq \|a - p_i\|$.

Since b is in S_i and $a' := a(b')$ is in R_{1-i}, we have $\|a' - b\| \geq s$. Hence, $\delta' = \|b' - a'\| \geq \|a' - b\| - \|b' - b\| \geq s - \|a - p_i\| = \delta$, as desired. $\square$

The rest of the argument is similar to what we already saw in Section 3 (and even simpler because we do not have case (A) this time): Starting at $b_0 \in S \setminus R$ such that $s(b_0) < h + \varepsilon$, where ε is as in Lemma A.6, we define b_{t+1}, for each $t \in \mathbb{N}$, to be the point b' corresponding to $b = b_t$. By the lemma, $s(b_t)$ always decreases by at least $\delta(b_0)$, leading to a contradiction. This proves Theorem 1.3.

Appendix B. Proofs of Theorem 2.1

There are two proofs of Theorem 2.1 available; we sketch the main ideas for the reader's convenience.

The *first proof*, from [2], doesn't establish the theorem in full generality—it works only for closed and disjoint sites in a Euclidean space, or more generally, in a finite-dimensional normed space with a rotund norm. In this proof, we build a sequence of inner approximations to $\mathbf{R}$ and outer approximations to $\mathbf{S}$. Namely, we set $\mathbf{R}^{(0)} := \mathbf{P}$, $\mathbf{S}^{(0)} := \mathbf{Dom}\,\mathbf{R}^{(0)}$ (this is the classical Voronoi diagram of the sites $P_1, \ldots, P_n$), and for $k = 1, 2, \ldots$ we put $\mathbf{R}^{(k)} := \mathbf{Dom}\,\mathbf{S}^{(k-1)}$, $\mathbf{S}^{(k)} := \mathbf{Dom}\,\mathbf{R}^{(k-1)}$.

Antimonotonicity of $\mathbf{Dom}$ and induction yield $\mathbf{R}^{(0)} \preceq \mathbf{R}^{(1)} \preceq \mathbf{R}^{(2)} \preceq \cdots$ and $\mathbf{S}^{(0)} \succeq \mathbf{S}^{(1)} \succeq \mathbf{S}^{(2)} \succeq \cdots$, as well as $\mathbf{R}^{(k)} \preceq \mathbf{S}^{(k)}$ for all k. We then define $\mathbf{R}$ and $\mathbf{S}$ by

$$R_i := \overline{\bigcup_{k=0}^{\infty} R_i^{(k)}}, \qquad S_i := \bigcap_{k=0}^{\infty} S_i^{(k)}.$$

It remains to show that $\mathbf{R}$ and $\mathbf{S}$ are as required. This is done in [2] for the case of point sites in $\mathbb{R}^2$ with the Euclidean norm. By inspecting the proof (Lemma 5.1 of [2]), we see that it uses only the following property of the underlying metric space (stated there as Lemma 3.1): *If P is a closed set, $X_1 \supseteq X_2 \supseteq \cdots$ is a decreasing sequence of closed sets with $X_1 \cap P = \emptyset$, and $X := \bigcap_{k=1}^{\infty} X_k$, then* $\mathrm{dom}(P, X) \subseteq \overline{\bigcup_{k=1}^{\infty} \mathrm{dom}(P, X_k)}$. (Moreover, in the proof one also needs that $P_i \cap S_j^{(0)} = \emptyset$ for $i \neq j$; since we assume the sites to be closed and disjoint, this property of the Voronoi diagram is immediate.)

To verify the above statement, we can again follow the proof of Lemma 3.1 in [2]. First we check that with the X_k as above and any point y, we have $\mathrm{dist}(y, X) = \lim_{k\to\infty} \mathrm{dist}(y, X_k)$; this follows easily assuming compactness of all closed balls in a finite-dimensional normed space. Now let us fix $x \in \mathrm{dom}(P, X)$ arbitrarily (we may assume $x \notin P$, since the case $x \in P$ is clear) and choose $\varepsilon > 0$; we want to show that $\mathrm{dist}(x, \mathrm{dom}(P, X_k)) \leq \varepsilon$ for some k. We let p be a point of P nearest to x, and choose a point $y \neq x$ on the segment px at distance smaller than ε from x. It is easy to check, using the rotundity of the norm, that $\mathrm{dist}(y, p) < \mathrm{dist}(y, X)$, and thus $\mathrm{dist}(y, p) \leq \mathrm{dist}(y, X_k)$ for k sufficiently large. So $y \in \mathrm{dom}(P, X_k)$ and we are done.

The *second proof* of Theorem 2.1, due to Reem and Reich [8], is based on the following theorem of Knaster and Tarski (see [9]): *If $\mathcal{L} = (L, \preceq)$ is a complete lattice and $g\colon \mathcal{L} \to \mathcal{L}$ is a monotone mapping, then g has at least one fixed point (i.e., $x \in L$ with $g(x) = x$), and moreover, there exists a smallest fixed point x_0 and a largest fixed point x_1, i.e., such that $x_0 \preceq x \preceq x_1$ for every fixed point x.* To prove Theorem 2.1, we let L be the system of all ordered n-tuples $\mathbf{D}$ such that $P_i \subseteq D_i$ for every i. We introduce the ordering $\preceq$ as above (one has to check that this gives a complete lattice, which is straightforward). Let $g := \mathbf{Dom}^2$; that is, $g(\mathbf{D}) := \mathbf{Dom}(\mathbf{Dom}\,\mathbf{D})$. Then we let $\mathbf{R}$ be the smallest fixed point of g as in the Knaster–Tarski theorem, and $\mathbf{S} := \mathbf{Dom}\,\mathbf{R}$. Clearly $\mathbf{Dom}\,\mathbf{S} = \mathbf{Dom}^2\,\mathbf{R} = g(\mathbf{R}) = \mathbf{R}$. Moreover, if $\mathbf{R}', \mathbf{S}'$ satisfy $\mathbf{R}' = \mathbf{Dom}\,\mathbf{S}'$ and $\mathbf{S}' = \mathbf{Dom}\,\mathbf{R}'$, then $\mathbf{R}'$ and $\mathbf{S}'$ are both fixed points of $\mathbf{Dom}^2$, and thus $\mathbf{R} \preceq \mathbf{R}', \mathbf{S}' \preceq \mathbf{S}$ as claimed.

Appendix C. Proof of Lemma 5.2

The construction has two positive parameters, α and δ, where α is small and δ is still much smaller.

We let $\|\cdot\|'$ be the Euclidean norm scaled by α in the horizontal direction; that is, $\|(x, y)\|' = \sqrt{\alpha^2 x^2 + y^2}$. Let $\|\cdot\|''$ be the ℓ_1 norm scaled by a suitable factor β (close to 1) in the vertical

direction: $\|(x,y)\|'' = |x| + \beta|y|$. The norm $\|\cdot\|_{(1)}$ is obtained as $a'\|\cdot\|' + a''\|\cdot\|''$, where $a', a'' > 0$ are suitable coefficients. This obviously yields a norm, which is rotund since $\|\cdot\|'$ is rotund.

We want that the contribution of $\|\cdot\|'$ is small compared to that of $\|\cdot\|''$, and that the corners of the unit ball of $\|\cdot\|_{(1)}$ coincide with those of the ℓ_1 unit ball. This finally leads to the formula

$$\|(x,y)\|_{(1)} := \delta\sqrt{\alpha^2x^2 + y^2} + (1 - \alpha\delta)|x| + (1 - \delta)|y|.$$

Fig. 11 is actually obtained from this formula with $\delta = 0.7$ and $\alpha = 0.5$. It is easy to check that, as the picture suggests, $\|\cdot\|_{(1)} \leq \|\cdot\|_1$ (and thus the ℓ_1 unit ball is contained in the $\|\cdot\|_{(1)}$ unit ball), and for δ is sufficiently small in terms of α and ε, the unit ball of $\|\cdot\|_{(1)}$ is contained in the octagon as in the lemma.

It remains to investigate the bisector of p and q for $x \geq 1$ and $y \geq 1$. For convenience, we translate p and q by $(-1,-1)$ and scale by $\frac{1}{2}$. Then the bisector is given by the equation $\|(x+1,y)\|_{(1)} = \|(x,y+1)\|_{(1)}$, with the region of interest being the positive quadrant $x, y \geq 0$. For $x, y \geq 0$, the absolute values can be removed, δ disappears from the equation, and we obtain $\sqrt{\alpha^2(x+1)^2 + y^2} + 1 - \alpha = \sqrt{\alpha^2x^2 + (y+1)^2}$. This can be solved for y explicitly, with the only positive root

$$y = \frac{1-\alpha}{2-\alpha}\left(\sqrt{1 + 2\alpha x + 2\alpha x^2} - 1 + \frac{\alpha}{1-\alpha}x\right).$$

This is the equation of the bisector curve in the positive quadrant. It is a simple exercise in calculus (distinguishing the cases $\alpha x \leq 1$ and $\alpha x > 1$, say) to show that $y \leq C\sqrt{\alpha}\,x$ for all $x > 0$ and all sufficiently small α (here C is a suitable constant).